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Particular solution of DGLAP evolution equation in next-to leading order and x-distributions of deuteron structure functions at low-x

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Abstract : We present particular solutions of singlet and non-singlet Dokshitzer-Gribov Lipatov-Altarelli-Parisi (DGLAP) evolution equations in next-to-leading order (NLO) at low-x. We obtain x-evolutions of deuteron structure functions at low-x from DGLAP evolution equations. The results of x-evolutions are compared with NMC low-x and low-Q^2 data and with those of leading order (LO) solutions of DGLAP evolution equations

Keywords : Dokshitzer-Gribov-Lipatov-Altarelli Parisi evolution equations, particular solution, deuteron structure functions

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1. Introduction

The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [1-4] are fundamental tools to study the $t\left(=\ln\left(Q^2/\Lambda^2\right)\right)$ and x-evolutions of structure functions, where x and Q^2 are Bjorken variable and four momenta transfer respectively in a deep inelastic scattering (DIS) process [5] and Λ is the QCD cut-off parameter. On the other hand, the study of structure functions at low-x has become topical in view [6] of high energy collider and super collider experiments [7]. Solutions of DGLAP evolution equations give quark and gluon structure functions which ultimately produce, proton, neutron and deuteron structure functions. Those structure functions are important inputs in many high energy processes. Moreover the determination of their t and x-evolutions is a test for QCD, the underlying dynamics of quarks and gluons inside hadrons. Though some numerical solutions are available in the literature [8, 9], the explorations of the possibility of obtaining analytical solutions of DGLAP evolution equations are always interesting. In this connection, particular solutions of DGLAP evolution equations at low-x in leading order (LO) have already been obtained by applying Taylor expansion method [10] and t and x-evolutions [11-15] of structure functions for intermediate and low-x have been presented. Here, the particular solutions have been obtained either by a linear combination of U and V of the general solution $f(U, V) = 0$ [11-13] or from the complete solution [14, 15] of the equation. We also have obtained particular solution of DGLAP evolution equation from the complete solution in next-to-leading order (NLO) for non-singlet and singlet structure functions [15, 16] and compared our results with HERA H1 [17] and NMC [18] data.

The present paper reports particular solutions of DGLAP evolution equations computed from complete solutions in NLO at low-x and calculation of t and x-evolutions for singlet and non-singlet structure functions, and hence x-evolutions of deuteron structure functions. In some instance, we can deal with particular solutions more conveniently than with the general solutions [19]. In calculating structure functions, input data points have been taken from experimental data directly unlike the usual practice of using an input distribution function introduced by hand. These NLO results are compared with the NMC low-x, low-Q^2 data and with those of particular solution in LO. Here, Section 1, Section 2, and Section 3 present the introduction, the relevant theory and the results and discussion, respectively.

2. Theory

Though the basic theory has been discussed elsewhere [15,

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16], here we have mentioned some essential steps for clarity. The DGLAP evolution equations with splitting functions [20, 21] for singlet and non-singlet structure functions in NLO are in the standard forms [22]

$$\frac{\partial F_2^S(x,t)}{\partial t}-\frac{\alpha_s(t)}{2\pi}$$
$$\times\left[\frac{2}{3}\{3+4\ln(1-x)\}F_2^S(x,t)+\frac{4}{3}\int_x^1\frac{dw}{1-w}\left\{(1+w^2)F_2^S\left(\frac{x}{w},t\right)\right.\right.$$
$$\left.\left.-2F_2^S(x,t)\right\}+N_f\int_x^1\left\{w^2+(1-w^2)\right\}G\left(\frac{x}{w},t\right)\right]-\left(\frac{\alpha_s(t)}{2\pi}\right)^2$$
$$\times\left[(x-1)F_2^S(x,t)\int_0^1 f(w)dw+\int_x^1 f(w)F_2^S\left(\frac{x}{w},t\right)dw\right.$$
$$\left.+\int_x^1 F_{qq}^S(w)F_2^S\left(\frac{x}{w},t\right)dw\right]+\left(\frac{\alpha_s(t)}{2\pi}\right)^2\int_x^1 F_{qg}^S(w)G\left(\frac{x}{w},t\right)dw=0 \tag{1}$$

and

$$\frac{\partial F_2^{NS}(x,t)}{\partial t}-\frac{\alpha_s(t)}{2\pi}$$
$$\times\left[\frac{2}{3}\{3+4\ln(1-x)\}F_2^{NS}(x,t)+\frac{4}{3}\int_x^1\frac{dw}{1-w}\left\{(1+w^2)F_2^{NS}\left(\frac{x}{w},t\right)\right.\right.$$
$$\left.\left.-2F_2^{NS}(x,t)\right\}\right]-\left(\frac{\alpha_s(t)}{2\pi}\right)^2\left[(x-1)F_2^{NS}(x,t)\int_0^1 f(w)dw\right.$$
$$\left.+\int_x^1 f(w)F_2^{NS}\left(\frac{x}{w},t\right)dw\right]=0, \tag{2}$$

where $\beta_0=\dfrac{33-2N_f}{3}$ and $\beta_1=\dfrac{306-38N_f}{3}$, N_f being the number of flavours.

Here, $f(w)=C_F^2\left[P_F(w)-P_A(w)\right]+\frac{1}{2}C_FC_A\left[P_G(w)+P_A(w)\right]$
$+C_FT_RN_fP_{N_f}(w)$,

$$F_{qq}^S(w)=2C_FT_RN_fF_{qq}(w)$$

and $\quad F_{qg}^S(w)=C_FT_RN_fF_{qg}^1(w)+C_GT_RN_fF_{qg}^2(w)$.

The explicit forms of higher order kernels are [20–21] :

$$P_F(w)=-2\left(\frac{1+w^2}{1-w}\right)\ln w\ln(1-w)-\left(\frac{3}{1-w}+2w\right)\ln w$$
$$-\frac{1}{2}(1+w)\ln^2 w-5(1-w),$$

$$P_G(w)=\frac{1+w^2}{1-w}\left(\ln^2 w+\frac{11}{3}\ln w+\frac{67}{9}-\frac{\pi^2}{3}\right)$$
$$+2(1+w)\ln w+\frac{40}{3}(1-w),$$

$$P_{N_f}(w)=\frac{2}{3}\left[\frac{1+w^2}{1-w}\left(-\ln w-\frac{5}{3}\right)-2(1-w)\right],$$

$$P_A(w)=2\left(\frac{1+w^2}{1-w}\right)\int_{w/(1+w)}^{1/(1+w)}\frac{dk}{k}\ln\frac{1-k}{k}+2(1+w)\ln w+4(1-w),$$

$$F_{qq}(w)=\frac{20}{9w}-2+6w-\frac{56}{9}w^2+\left(1+5w+\frac{8}{3}w^2\right)\ln w$$
$$-(1+w)\ln^2 w,$$

$$F_{qg}^1(w)=4-9w-(1-4w)\ln w-(1-2w)\ln^2 w+4\ln(1-w)$$
$$+\left[2\ln^2\left(\frac{1-w}{w}\right)-4\ln\left(\frac{1-w}{w}\right)-\frac{2}{3}\pi^2+10\right]P_{qg}(w)$$

and

$$F_{qg}^2(w)=\frac{182}{9}+\frac{14}{9}w+\frac{40}{9w}+\left(\frac{136}{3}w-\frac{38}{3}\right)\ln w-4\ln(1-w)$$
$$-(2+8w)\ln^2 w+\left[-\ln^2 w+\frac{44}{3}\ln w-2\ln^2(1-w)+4\ln(1-w)\right.$$
$$\left.+\frac{\pi^2}{3}-\frac{218}{3}\right]P_{qg}(w)+2P_{qg}(-w)\int_{w/(1+w)}^{1/(1+w)}\frac{dz}{z}\ln\frac{1-z}{z},$$

where $P_{qg}(w)=w^2+(1-w)^2$, $C_A=C_G=N_C=3$,

$C_F=(N_C^2-1)/2N_C$ and $T_R=1/2$.

Let us introduce the variable $u=1-w$ and note that [23]

$$\frac{x}{w}=\frac{x}{1-u}=x\sum_{k=0}^{\infty}u^k. \tag{3}$$

The series (3) is convergent for $|u|<1$. Since $x<w<1$, so $0<u<1-x$ and hence the convergence criterion is satisfied. Now, using Taylor expansion method [10], we can rewrite $F_2^S(x/w,t)$ as

$$F_2^S(x/w,t)=F_2^S\left(x+x\sum_{k=1}^{\infty}u^k,t\right)$$

$$F_2^S(x,t) + x\sum_{k=1} u^k \frac{\partial F_2^S(x,t)}{\partial x} + \frac{1}{2}x^2\left(\sum_{k=1}^{\infty} u^k\right)^2 \frac{\partial^2 F_2^S(x,t)}{\partial x^2} + \ldots \tag{4}$$

which covers the whole range of u, $0 < u < 1-x$. Since x is small in our region of discussion, the terms containing x^2 and higher powers of x can be neglected as our first approximation as discussed in our earlier works [11, 12, 14–16]. $F_2^S(x/w,t)$ can be approximated for small-x as

$$F_2^S(x/w,t) \cong F_2^S(x,t) + x\sum_{k=1}^{\infty} u^k \frac{\partial F_2^S(x,t)}{\partial x}. \tag{5}$$

Similarly, $G(x/w,t)$ and $F_2^{NS}(x/w,t)$ can be approximated for small-x as

$$G(x/w,t) \cong G(x,t) + x\sum_{k=1}^{\infty} u^k \frac{\partial G(x,t)}{\partial x} \tag{6}$$

and

$$F_2^{NS}(x/w,t) \cong F_2^{NS}(x,t) + x\sum_{k=1} u^k \frac{\partial F_2^{NS}(x,t)}{\partial x} \tag{7}$$

Using eq. (3), (5) and (6) in eq. (1) and performing integrations, we get

$$\begin{aligned}
&\frac{\partial F_2^S(x,t)}{\partial t} - \left[\frac{\alpha_s(t)}{2\pi}A_1(x) + \left(\frac{\alpha_s(t)}{2\pi}\right)^2 B_1(x)\right] F_2^S(x,t) \\
&- \left[\frac{\alpha_s(t)}{2\pi}A_2(x) + \left(\frac{\alpha_s(t)}{2\pi}\right)^2 B_2(x)\right] G(x,t) \\
&- \left[\frac{\alpha_s(t)}{2\pi}A_3(x) + \left(\frac{\alpha_s(t)}{2\pi}\right)^2 B_3(x)\right] \frac{\partial F_2^S(x,t)}{\partial x} \\
&- \left[\frac{\alpha_s(t)}{2\pi}A_3(x) + \left(\frac{\alpha_s(t)}{2\pi}\right)^2 B_3(x)\right] \\
&\frac{\alpha_s(t)}{2\pi}A_4(x) + \left(\frac{\alpha_s(t)}{2\pi}\right)^2 B_4(x) \frac{\partial G(x,t)}{\partial x} = 0,
\end{aligned} \tag{8}$$

where

$$A_1(x) = \frac{2}{3}\{3 + 4\ln(1-x) + (x-1)(x+3)\},$$

$$A_2(x) = N_f\left[\frac{1}{3}(1-x)\left(2 - x + 2x^2\right)\right],$$

$$A_3(x) = \frac{2}{3}\left\{x\left(1-x^2\right) + 2x\ln\left(\frac{1}{x}\right)\right\},$$

$$A_4(x) = N_f x\left\{\ln\frac{1}{x} - \frac{1}{3}(1-x)\left(5 - 4x + 2x^2\right)\right\},$$

$$\begin{aligned}
B_1(x) = {} & x\int_0^1 f(w)dw - \int_0^x f(w)dw + \frac{4}{3}N_f\left[-\ln x\left(\frac{20}{9} + 3x + 3x^2\right.\right. \\
& \left.\left. + \frac{8}{9}x^3\right) + \frac{1}{2}x(2+x)\ln^2 x + 5x - \frac{3}{2}x^2 + \frac{64}{27}x^3 - \frac{317}{54}\right],
\end{aligned}$$

$$\begin{aligned}
B_2(x) = {} & \frac{2}{3}N_f\left[-\left(9x - 5x^2 + \frac{32}{9}x^3\right)\ln x\right. \\
& + \left(3x - 3x^2 + \frac{4}{3}x^3\right)\ln^2 x + \left(\frac{32}{9}x^3 - \frac{14}{3}x^2 + \frac{8}{3}x - \frac{14}{9}\right)\ln(1-x) \\
& + \left(\frac{4}{3} - 2x + 2x^2 - \frac{4}{3}x^3\right)\ln^2(1-x) + \left(\frac{2}{3} - \frac{4}{9}\pi^2\right. \\
& \left. + \left(\frac{2}{3}\pi^3 - \frac{59}{9}\right)x + \left(\frac{113}{9} - \frac{2}{3}\pi^2\right)x^2 + \left(\frac{4}{9}\pi^2 - \frac{20}{3}\right)x^3\right] \\
& + \frac{3}{2}N_f\left[-\left(\frac{40}{9} + 8x + 11x^2 + \frac{92}{9}x^2\right)\ln x\right. \\
& + \left(3x + 3x^2 + \frac{2}{9}x^3\right)\ln^2 x + \left(\frac{14}{9} - \frac{8}{3}x + \frac{14}{3}x^2 - \frac{32}{9}x^3\right)\ln(1-x) \\
& + \left(-\frac{4}{3} + 2x - 2x^2 + \frac{4}{3}x^3\right)\ln^2(1-x) + \left(\frac{2}{9}\pi^2 - \frac{769}{54}\right) \\
& + \left(\frac{122}{9} - \frac{1}{3}\pi^2\right)x + \left(\frac{1}{3}\pi^2 - \frac{361}{18}\right)x^2 + \left(\frac{560}{27} - \frac{2}{9}\pi^2\right)x^3 \\
& + \frac{3}{2}N_f\int_x^1 2\left(w^2 + (1+w)^2\right)\left[-\ln\ln(1+w) - \ln w\ln(1+w)\right. \\
& \left.\left. + \ln\ln\left(\frac{1+w}{w}\right) + \ln\frac{1}{w}\ln\left(1 + \frac{1}{w}\right)\right]\right.
\end{aligned}$$

$$\begin{aligned}
B_3(x) = {} & x\int_x^1 \frac{1-w}{w} f(w)dw \\
& + \frac{4}{3}N_f x\left[\left(\frac{38}{9} - 4x + \frac{5}{3}x^2 + \frac{8}{9}x^3\right)\ln x - \frac{1}{2}\left(1 + x^2\right)\ln^2\right. \\
& \left. + \frac{1}{3}\ln^3 x + \frac{38}{9x} - 4x + \frac{95}{18}x^2 - \frac{64}{27}x^3 - \frac{61}{54}\right],
\end{aligned}$$

and

$$B_4(x) = x\int_x^1 \frac{1-w}{w} F_{qg}^S(w)dw.$$

Let us assume for simplicity [11-14],

$$G(x,t) = K(x)\, F_2^S(x,t), \tag{9}$$

where $K(x)$ is a function of x. Now eq. (8) becomes

$$\frac{\partial F_2^S(x,t)}{\partial t} - \frac{\alpha_s(t)}{2\pi} L_1(x) + \frac{\alpha_s(t)}{2\pi} M_1(x)\, F_2^S(x,t)$$

$$\frac{\alpha_s(t)}{2\pi} L_2(x) + \left(\frac{\alpha_s(t)}{2\pi}\right)^2 M_2(x)\, \frac{\partial F_2^S(x,t)}{\partial x} = 0, \tag{10}$$

where

$$L_1(x) = A_1(x) + K(x)A_2(x) + A_4(x)\frac{\partial K(x)}{\partial x},$$

$$M_1(x) = B_1(x) + K(x)B_2(x) + B_4(x)\frac{\partial K(x)}{\partial x},$$

$$L_2(x) = A_3(x) + K(x)A_4(x).$$

and $M_2(x) = B_3(x) + K(x)B_4(x)$.

For a possible solution, we assume [15, 22] that

$$\left(\frac{\alpha_s(t)}{2\pi}\right)^2 = T_0\left(\frac{\alpha_s(t)}{2\pi}\right) \tag{11}$$

where T_0 is a numerical parameter to be obtained from the particular Q^2-range under study. By a suitable choice of T_0, we can reduce the error to a minimum. Now eq. (10) can be recast as

$$\frac{\partial F_2^S(x,t)}{\partial t} - P_S(x,t)\frac{\partial F_2^S(x,t)}{\partial x} - Q_S(x,t)F_2^S(x,t) = 0, \tag{12}$$

where $P_S(x,t) = \frac{\alpha_s(t)}{2\pi}\left[L_2(x) + T_0 M_2(x)\right]$

and $Q_S(x,t) = \frac{\alpha_s(t)}{2\pi}\left[L_1(x) + T_0 M_1(x)\right]$.

Secondly, using eqs. (3), (7) and (11) in eq. (2) and performing u-integration, we have

$$\frac{\partial F_2^{NS}(x,t)}{\partial t} - P_{NS}(x,t)\frac{\partial F_2^{NS}(x,t)}{\partial x} - Q_{NS}(x,t)F_2^{NS}(x,t) = 0, \tag{13}$$

where $P_{NS}(x,t) = \frac{\alpha_s(t)}{2\pi}\left[A_5(x) + T_0 B_5(x)\right]$

and $Q_{NS}(x,t) = \frac{\alpha_s(t)}{2\pi}\left[A_6(x) + T_0 B_6(x)\right]$

with $A_5(x) = \frac{2}{3}\left\{x\left(1-x^2\right) + 2x\ln\left(\frac{1}{x}\right)\right\}$,

$$B_5(x) = x\int_x^1 \frac{1-w}{w} f(w)dw,$$

$$A_6(x) = \frac{4}{3}\{3 + 4\ln(1-x) + (x-1)(x+3)\},$$

and $B_6(x) = -\int_0^x f(w)dw + x\int_0^x f(w)dw$.

The general solutions [10, 19] of eqs. (12) is $F(U, V) = 0$, where F is an arbitrary function and $U\left(x, t, F_2^S\right) = C_1$ and $V\left(x, t, F_2^S\right) = C_2$ where, C_1 and C_2 are constants and they form a solution of equations

$$\frac{dx}{P_S(x,t)} = \frac{dt}{-1} = \frac{dF_2^S(x,t)}{-Q_S(x,t)} \tag{14}$$

We observed that the Lagrange's auxiliary system of ordinary differential equations [10, 19] occurred in the formalism, can not be solved without the additional assumption of linearization (eq. (11)) and introduction of an *ad hoc* parameter T_0. Which does not affect the results of t-evolution of structure functions. Solving eq. (14), we obtain

$$U\left(x, t, F_2^S\right) = t^{(b/t+1)}\exp\left[\frac{b}{t} + \frac{N_S(x)}{a}\right]$$

and $V\left(x, t, F_2^S\right) = F_2^S(x,t)\exp\left[M_S(x)\right]$,

where $a = \frac{2}{\beta_0}$, $b = \frac{\beta_1}{\beta_0^2}$, $N_S(x) = \int \frac{dx}{L_2(x) + T_0 M_2(x)}$

and $M_S(x) = \int \frac{L_1(x) + T_0 M_1(x)}{L_2(x) + T_0 M_2(x)}dx$. If U and V are two independent solutions of eq. (14) and if α and β are arbitrary constants, then $V = \alpha U + \beta$ may be taken as a complete solution of eq. (14). We take this form as this is the simplest form of a complete solution which contains both the arbitrary constants α and β. Earlier [11, 12], we considered an equation $AU + BV = 0$, where A and B are arbitrary constants. But that is not a complete solution having both the arbitrary constants as this equation can be transformed to the form $V = CU$, where $C = -A/B$, *i.e.* the equation contains only one arbitrary constant. Then the complete solution [10, 19]

$$F_2^S(x,t)\exp\left[M_S(x)\right] = \alpha\left[t^{(b/t+1)}\exp\left(\frac{b}{t} + \frac{N_S(x)}{a}\right)\right] + \beta \tag{15}$$

is a two-parameter family of planes which does not have an envelope, since the arbitrary constants enter linearly [10]. Again, differentiating eq. (15) with respect to β, we get $0 = 1$, which is absurd. Hence, there is no singular solution. The one parameter family determined by taking $\beta = \alpha^2$ has equation

$$F_2^S(x,t)\exp\left[M_S(x)\right] = \alpha\left[t^{(b/t+1)}\exp\left(\frac{b}{t} + \frac{N_S(x)}{a}\right)\right] + \alpha^2. \tag{16}$$

Differentiating eq. (16) with respect to α, we obtain

$$\alpha = -\frac{1}{2}t^{(b/t+1)}\exp\left[\frac{b}{t}+\frac{N_S(x)}{a}\right]$$

Putting the value of α again in equation (16), we obtain the envelope

$$F_2^S(x,t)\exp[M_S(x)] = -\frac{1}{4}\left[t^{(b/t+1)}\exp\left(\frac{b}{t}+\frac{N_S(x)}{a}\right)\right]^2.$$

Therefore,

$$F_2^S(x,t) = -\frac{1}{4}t^{2(b/t+1)}\exp\left[\frac{2b}{t}+\frac{2N_S(x)}{a}-M_S(x)\right], \quad (17)$$

which is merely a particular solution

Now, defining

$$F_2^S(x,t_0) = -\frac{1}{4}t_0^{2(b/t_0+1)}\exp\left[\frac{2b}{t_0}+\frac{2N_S(x)}{a}-M_S(x)\right].$$

at $t = t_0$, where $t_0 = \ln\left(Q_0^2/\Lambda^2\right)$ at any lower value $Q = Q_0$, we get from eq. (17),

$$F_2^S(x,t) = F_2^S(x,t_0)\left|\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right|\exp\left|2b\left|\frac{1}{t}-\frac{1}{t_0}\right.\right. \quad (18)$$

which gives the t-evolution of singlet structure function $F_2^S(x,t)$ in NLO for $\beta = \alpha^2$.

Proceeding exactly in the same way, and defining

$$F_2^{NS}(x,t_0) = -\frac{1}{4}t_0^{2(b/t_0+1)}\exp\left[\frac{2b}{t_0}+\frac{2N_{NS}(x)}{a}-M_{NS}(x)\right]$$

where $N_{NS}(x) = \int \frac{dx}{A_5(x)+T_0B_5(x)}$

and $M_{NS}(x) = \int \frac{A_6(x)+T_0B_6(x)}{A_5(x)+T_0B_5(x)}dx$,

we get for non-singlet structure function in NLO as

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)\exp\left|2b\left|\frac{1}{t}-\frac{1}{t_0}\right.\right. \quad (19)$$

which gives the t-evolution of non-singlet structure function $F_2^{NS}(x,t)$ in NLO for $\beta = \alpha^2$.

In an earlier communication [14], we suggested that for low x in LO for $\beta = \alpha^2$,

$$F_2^S(x,t) = F_2^S(x,t_0 \quad (20)$$

and

$$F_2^{NS}(x,t) - F_2^{NS}(x,t_0)\Big|_{t_0}^{t} \quad (21)$$

We observe that if b tends to zero, eqs. (18) and (19) tend to eqs. (20) and (21), respectively, *i.e.*, solutions of NLO equations go over to those of LO equations. Physically, b tends to zero means number of flavours is high.

Again defining

$$F_2^S(x_0,t) = -\frac{1}{4}t^{(b/t+1)}\exp\left[\frac{2b}{t}+\frac{2N_S(x)}{a}-M_S(x)\right.$$

we obtain from eq. (17)

$$F_2^S(x,t) = F_2^S(x_0,t)\exp\int\left[\frac{2}{a}\frac{1}{L_2(x)+T_0M_2(x}\right.$$

$$\left.\frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right]dx, \quad (22)$$

which gives the x-evolution of singlet structure function $F_2^S(x,t)$ in NLO for $\beta = \alpha^2$. Similarly, defining $F_2^{NS}(x_0,t) = -\frac{1}{4}t^{(b/t+1)}$ $\times\exp\left[\frac{2b}{t}+\frac{2N_{NS}(x)}{a}-M_{NS}(x)\right]_{x=x_0}$, we get

$$F_2^{NS}(x,t) = F_2^{NS}(x_0,t)\exp\int_{x_0}\left|\frac{2}{a}\frac{}{A_5(x)+T_0B_5(x)}\right.$$

$$\left.\frac{A_6(x)+T_0B_6(x)}{A_5(x)+T_0B_5(x)}\right|dx, \quad (23)$$

which gives the x-evolution of non-singlet structure function $F_2^{NS}(x,t)$ in NLO for $\beta = \alpha^2$.

In an earlier communication [14], we suggested that for low-x in LO for $\beta = \alpha^2$,

$$F_2^S(x,t) = F_2^S(x_0,t)\exp\int_{x_0}^{x}\left(\frac{2}{A_f M(x)}\quad\frac{L(x)}{M(x)}\right)dx \quad (24)$$

and

$$F_2^{NS}(x,t) = F_2^{NS}(x_0,t)\exp\int_{x_0}^{x}\left(\frac{2}{A_f Q(x)}-\frac{L(x)}{M(x)}\right)dx, \quad (25)$$

where

$A_f = 4/(33-2N_f)$, $P(x) = 3+4\ln(1-x)-(1-x)(x+3)$,

$Q(x) = x(1-x^2)-2x\ln x$,

$$L(x) = P(x) + K(x)C(x) + D(x)\frac{\partial K(x)}{\partial x}$$

and $\quad M(x) = Q(x) + K(x)D(x)$, where again,

$$C(x) = 1/2N_f(1-x)(2-x+2x^2)$$

and $\quad D(x) = N_f x\left[-1/2(1-x)(5-4x+2x^2) + (3/2)\ln(1/x)\right].$

Of course, unlike for the t-evolution equations, we could not have for the x-evolution equations in LO as some limiting case of NLO equations. Deuteron, proton and neutron structure functions measured in deep inelastic electro-production can be written in terms of singlet and non-singlet quark distribution functions [5] as

$$F_2^d(x,t) = 5/9\, F_2^S(x,t), \tag{26}$$

$$F_2^p(x,t) = 5/18\, F_2^S(x,t) + 3/18\, F_2^{NS}(x,t) \tag{27}$$

and

$$F_2^n(x,t) = 5/18\, F_2^S(x,t) - 3/18\, F_2^{NS}(x,t). \tag{28}$$

Now using eqs. (22) in eq. (26), we will get x-evolution of deuteron structure function $F_2^S(x,t)$ at low-x in NLO for $\beta = \alpha^2$ as

$$F_2^d(x,t) = F_2^d(x_0,t)\exp\int_{x_0}^{x}\left[\frac{2}{a}\cdot\frac{1}{L_2(x)+T_0M_2(x)} - \frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right]dx, \tag{29}$$

where, the input function is $F_2^d(x_0,t) = \frac{5}{9}F_2^S(x_0,t)$. The corresponding result for a particular solution of DGLAP evolution equations in LO for $\beta = \alpha^2$ obtained earlier [14] is

$$F_2^d(x,t) = F_2^d(x_0,t)\exp\left[\int_{x_0}^{x}\left(\frac{}{A_f M(x)} - \frac{L(x)}{M(x)}\right)dx\right]. \tag{30}$$

The determination of x-evolutions of proton and neutron structure functions like those of deuteron structure function is not possible by this methodology, because to extract the x-evolution of protron and neutron structure functions we are to use eqs. (22) and (23) in eqs. (27) and (28). But the functions inside the integral sign of eqs. (22) and (23) are different and we need to separate the input functions $F_2^S(x_0,t)$ and $F_2^{NS}(x_0,t)$ from the data points to extract the x-evolutions of the proton and neutron structure functions, which may contain large errors.

For the complete solution of eq. (12), we take $\beta = \alpha^2$ in eq. (15). If we take $\beta = \alpha$ in eq. (15) and differentiate with respect to α as before, we get $0 = t^{(b/t+1)}\exp\left(\frac{b}{t} + \frac{N_s(x)}{a}\right) + 1$, from which we can not determine the value of α. But if we take $\beta = \alpha^2$ in eq. (15) and differentiate with respect to α, we get $\alpha = \sqrt{-\frac{1}{3}t^{(b/t+1)}\exp\left(\frac{b}{t} + \frac{N_s(x)}{a}\right)}$ which is imaginary. Putting this value of α in eq. (15), we get ultimately

$$F_2^S(x,t) = t^{(b/t+1)^{3/2}}\left\{\left(-\frac{1}{3}\right)^{1/2} + \left(-\frac{1}{3}\right)^{3/2}\right\}$$

$$\times\exp\left(\frac{b}{t} + \frac{N_S(x)}{a}\right)^{3/2} - M_S(x)$$

Now, defining

$$F_2^S(x,t_0) = t_0^{(b/t+1)^{3/2}}\left\{\left(-\frac{1}{3}\right)^{1/2} + \left(-\frac{1}{3}\right)^{3/2}\right\}$$

$$\times\exp\left(\frac{b}{t_0} + \frac{N_S(x)}{a}\right)^{3/2} - M_S(x)$$

we get

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{3/2}\exp\left[\frac{3}{2}b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]$$

Proceeding exactly in the same way, we also get for non-singlet structure function

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{3/2}\exp\left[\frac{3}{2}b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]$$

Then using eqs. (26), (27) and (28), we get t-evolutions of deuteron, proton and neutron structure functions

$$F_2^{d,p,n}(x,t) = F_2^{d,p,n}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{3/2}\exp\left[\frac{3}{2}b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]$$

Proceeding in the same way, we get x-evolution of deuteron structure function as

$$F_2^d(x,t) = F_2^d(x_0,t)\exp\int_{x_0}^{x}\left(\frac{3/2}{a}\cdot\frac{1}{L_2(x)+T_0M_2(x)} - \frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right)dx.$$

Similarly, we can show that if we take $\beta = \alpha^4$, we get

$$F_2^{d,p,n}(x,t) = F_2^{d,p,n}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{4/3}\exp\left[\frac{4}{3}b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]$$

and

$$F_2^d(x,t) = F_2^d(x_0,t)\exp\int_{x_0}^{x}\left(\frac{4/3}{a}\quad L_2(x)+T_0M_2(x) - \frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right)dx.$$

Similarly, if we take $\beta = \alpha^5$, we get

$$F_2^{d,p,n}(x,t) = F_2^{d,p,n}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{5/4}\exp\left[\frac{5}{4}b\left(\frac{1}{t}-\frac{1}{t_0}\right)\right]$$

and

$$F_2^d(x,t) = F_2^d(x_0,t)\exp\int_{x_0}^{x}\left(\frac{5/4}{a}\cdot\frac{1}{L_2(x)+T_0M_2(x)} \quad \frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right)dx.$$

and so on.

Thus, we observe that if we take $\beta = \alpha$ in eq. (15), we can not obtain the value of α and also the required solution. But if we take $\beta = \alpha^2, \alpha^3, \alpha^4, \alpha^5$ ... and so on, we see that the powers of $t^{(b/t+1)}/t_0^{(b/t_0+1)}$ and coefficient of $b(1/t-1/t_0)$ of exponential part in t-evolutions of deuteron, proton and neutron structure functions are 2, 3/2, 4/3, 5/4 .... and so on, respectively, as discussed above. Similarly, for x-evolutions of deuteron structure functions, we set that the numerators of the first term inside the integral sign are 2, 3/2, 4/3, 5/4 .... and so on, respectively, for the same values of α. Thus, we see that if in the relation $\beta = \alpha^y$, y varies between 2 to a maximum value, the powers of $t^{(b/t+1)}/t_0^{(b/t_0+1)}$, co-efficient of $t^{(b/t+1)}/t_0^{(b/t_0+1)}$ of exponential part in t-evolution and the numerator of the first term in the integral sign in x-evolution varies between 2 to 1. Then, it is understood that the solutions of eqs. (12) and (13) obtained by this methodology, are not unique and so the t-evolutions of deuteron, proton and neutron structure functions, and x-evolution of deuteron structure function obtained by this methodology, are not unique.

Thus by this methodology, instead of having a single solution,we arrive at a band of solutions, the range of these solutions being reasonably narrow.

3. Results and discussion

For a quantitative analysis of x-distributions of structure functions, we calculate the integrals that occurred in eq. (29) for $N_f = 4$. In this case, we neglect the first and second term of function $B_1(x)$ as x is small.

In Figure 1, we present our results of x-distribution of deuteron structure functions F_2^d from eq. (29) for $K(x) = ax^b$ (dashed lines) and for $K(x) = ce^{dx}$ (solid lines) in the relation $\beta = \alpha^y$ for y minimum (lower dashed and solid lines) and maximum (upper dashed and solid lines), where a, b, c and d are constants and for representative values of Q^2 given in each figure. We compare them with NMC deuteron low-x low-Q^2 data [18]. In each graph, the data point for x-value just below 0.1 had been taken as input $F_2^d(x_0,t)$. If we take $K(x) = ax^b$, then agreement of the result for y minimum with experimental data is found to be excellent at $a = 10$, $b = 0.016$. On the other hand, if we take $K(x) = ce^{-dx}$, then agreement of the results for y minimum with experimental data is found to be good at $c = 0.5$, $d = -3.8$. In this connection, earlier we observed [14] that agreement of the results with experimental data was excellent for $K(x) = 4.5$ (constant), $a = 4.5$, $b = 0.01$, $c = 5$, $d = 1$ for low-x in leading order and there was no significant difference between the results for

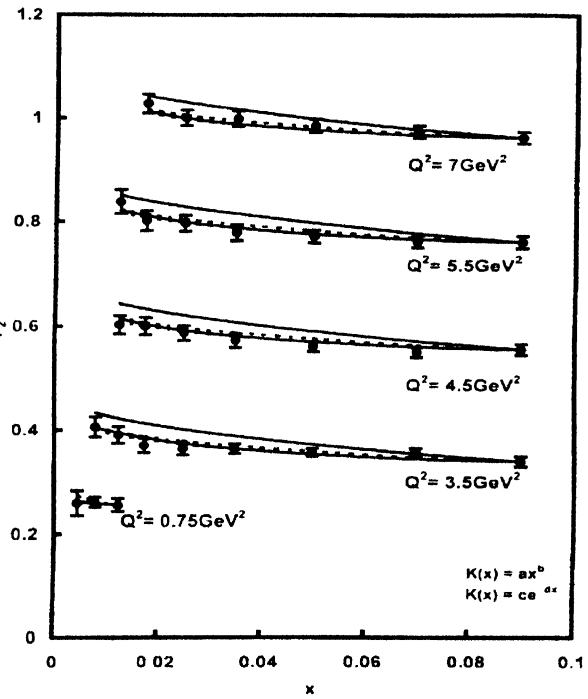

Figure 1. Results of x-distribution of deuteron structure functions F_2^d from eq. (29) for $K(x) = ax^b$ (dashed lines) and for $K(x) = ce^{-dx}$ (solid lines) in the relation for y minimum (lower dashed and solid lines) and maximum (upper dashed and solid lines), where a, b, c and d are constants and for representative values of Q^2 given in each figure, and compare them with NMC deuteron low-x low-Q^2 data [18]. In each graph, the data point for x-value just below 0.1 has been taken as input $F_2^d(x_0,t)$. If we take $K(x) = ax^b$ then agreement of the result for y minimum with experimental data is found to be excellent at $a = 10$, $b = 0.016$ On the other hand if we take $K(x) = ce^{-dx}$ then agreement of the results for y minimum with experimental data is found to be good at $c = 0.5$, $d = -3.8$ For convenience, value of each data point for one value of Q^2 is increased by adding 0.2i, where $i = 0, 1, 2, 3, \ldots$ are the numberings of curves counting from the bottom of the lowermost curve as the 0-th order.

y minimum and maximum in the relation $\beta = \alpha^{y}$. In the case of NLO, agreement of the results with experimental data is found to be very poor for any constant value of $K(x)$. Therefore, we do not present our result of x-distribution at $K(x)$ = constant in NLO.

In Figure 2, we present our results of x-evolution of deuteron structure function from eq. (29) for $K(x) = ax^b$ (dashed lines) and $K(x) = ce^{-dx}$ (solid lines) in the relation $\beta = \alpha^{y}$, for y maximum at different parameter values and for representative values of Q^2 given in each figure, and compare them with NMC deuteron low-x low-Q^2 data [18]. In each graph, the data point for x-value just below 0.1 has been taken as input $F_2^d(x_0, t)$. We observed that both the graphs coincide for each Q^2- value and are in excellent agreement with data when $a = 5.5$, $b = 0.016$, $c = 0.28$, $d = -3.8$.

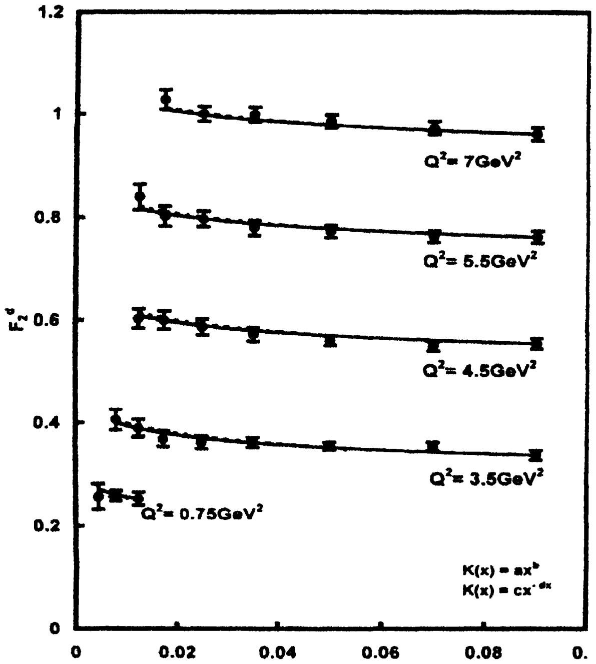

Figure 2. Results of x-evolution of deuteron structure function from equation (29) for $K(x) = ax^b$ (dashed lines) and $K(x) = ce^{-dx}$ (solid lines) in the relation $\beta = \alpha^{y}$, for y maximum at different parameter values and for representative values of Q^2 given in each figure, and compare them with NMC deuteron low-x low-Q^2 data [18]. In each graph, the data point for x-value just below 0.1 has been taken as input $F_2^d(x_0, t)$. We observed that both the graphs coincide for each Q^2-value and are in excellent agreement with data when $a = 5.5$, $b = 0.016$, $c = 0.28$, $d = -3.8$. For convenience, value of each data point for one value of Q^2 in increased by adding $0.2i$, where i = 0, 1, 2, 3, ... are the numberings of curves counting from the bottom of the lowermost curve as the 0-th order.

In Figure 3, we present the sensitivity of our results from eq. (29) for *a, b, c, d,* in the relation $\beta = \alpha^{y}$, for y minimum. In each graph (from top), if the absolute values of 'd', 'c', 'b', or 'a', respectively are increased, the curves shift upward and if the absolute values of 'd', 'c', 'b', *or* 'a', respectively are decreased, the curves move in the opposite direction. For the sensitivity of 'a', we take $b = 0.016$ and we observe that at $a = 10$, agreement of the results with experimental data is found to be excellent. For the sensitivity of 'b', we take $a = 10$ and we observe that at $b = 0.016$, agreement of the results with experimental data is found to be excellent. On the other hand for the sensitivity of 'c', we take $b = -3.8$ and we observe that at $c = 0.5$, agreement of the results with experimental data is found to be good. For the sensitivity of 'd', we take $c = 0.5$ and we observe that at $d = -3.8$, agreemnt of the results with experimental data is found to be excellent.

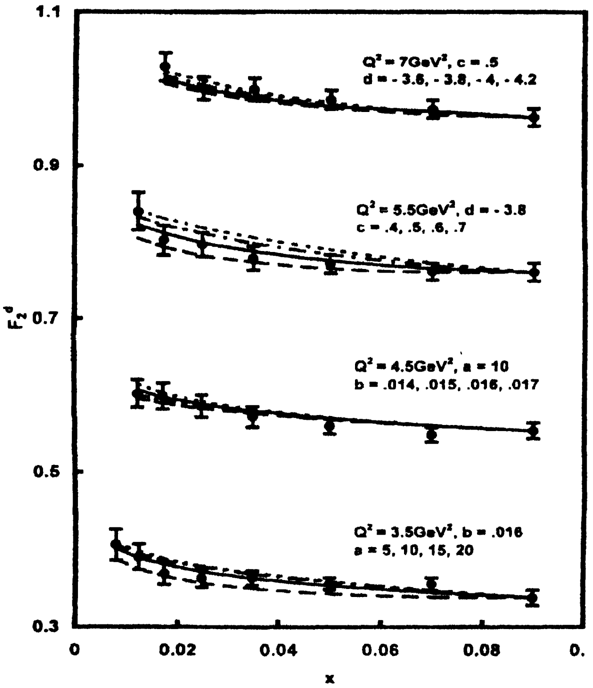

Figure 3. Sensitivity of our results of x-distribution of deuteron structure function in the relation $\beta = \alpha^{y}$ for y minimum for different values of a, b, c and d.

In Figure 4, we present the sensitivity of our results from eq. (29) for different values of 'T_0' at best fit of $K(x) = ax^b$ and $K(x) = ce^{-dx}$ in the realtion $\beta = \alpha^{y}$, for y minimum and for representative values of Q^2 given in each figure. Here, $a = 10$, $b = 0.016$, $c = 0.5$, $d = -3.8$. We observed that if the value of T_0 is increased, the curve moves slightly upward and if the value of T_0 is decreased, the curve moves slightly downward direction. But the nature of the curve remains some and difference between the curves are extremely small in both cases in the T_0 range mentioned in the figure.

In Figure 5, we present the results of x-evolution of deuteron structure function for $K(x) = ax^b$ (dashed lines) and $K(x) = ce^{-dx}$ (solid lines) in the relation $\beta = \alpha^{y}$, for y minimum in LO (lower dashed and solid lines) and in NLO (upper dashed and solid

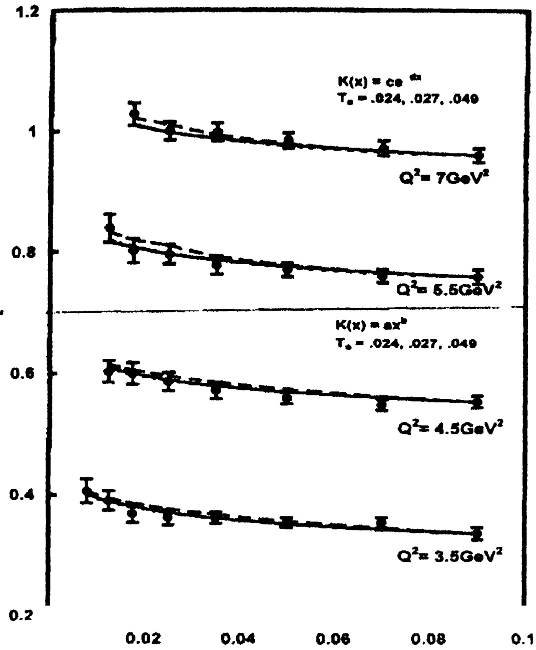

Figure 4. Sensitivity of our results of x-distribution of deuteron structure function in the relation $\beta = \alpha^{\nu}$ for v minimum for different values of a, b, c and d

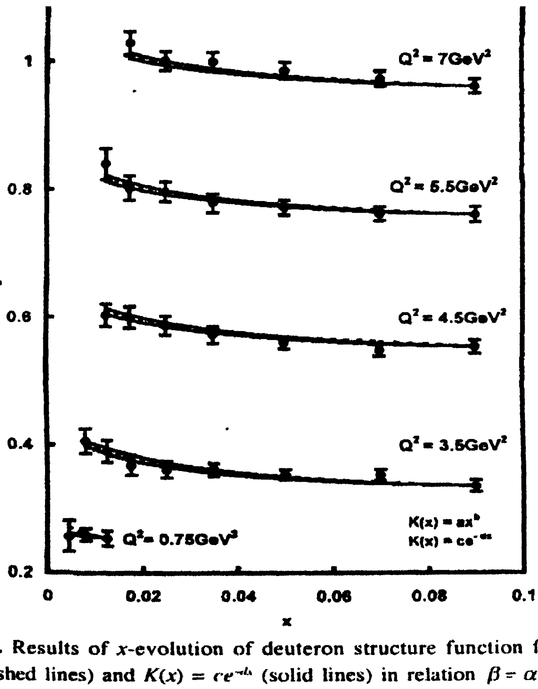

Figure 5. Results of x-evolution of deuteron structure function for $K(x) = ax^b$ (dashed lines) and $K(x) = ce^{-dx}$ (solid lines) in relation $\beta = \alpha^{\nu}$, for v minimum in LO (lower dashed and solid lines) and in NLO (upper dashed and solid lines) for representative values of Q^2 given in each figure, and compare them with NMC deuteron low-x low-Q^2 data [18]. In each graph, the data point for x-value just below 0.1 has been taken as input. Agreement of the result with experimental data is found to be excellent for $a = 4.5$, $b = 0.01$, $c = 5$, $d = 1$ in LO and $a = 10$, $b = 0.016$, $c = 0.5$, $d = -3.8$ in NLO and all the curves in each graph almost coincide.

lines) for representative values of Q^2 given in each figure, and compare them with NMC deuteron low-x low-Q^2 data [18]. In each graph, the data point for x-value just below 0.1 has been taken as input. Agreement of the result with experimental data is found to be excellent for $a = 4.5$, $b = 0.01$, $c = 5$, $d = 1$ in LO and $a = 10$, $b = 0.016$, $c = 0.5$, $d = -3.8$ in NLO and all curves in each graph almost coincide.

In Figure 6, we plot $T(t)^2$ (solid line) and $T_0T(t)$ (dashed line), where $T(t) = \alpha_s/2\pi$ against Q^2 in the Q^2 range $0.5 \le Q^2 \le 50$ GeV^2. We observed that for $T_0 = 0.027$, error becomes minimum in the Q^2 range $0.5 \le Q^2 \le 50$ GeV^2.

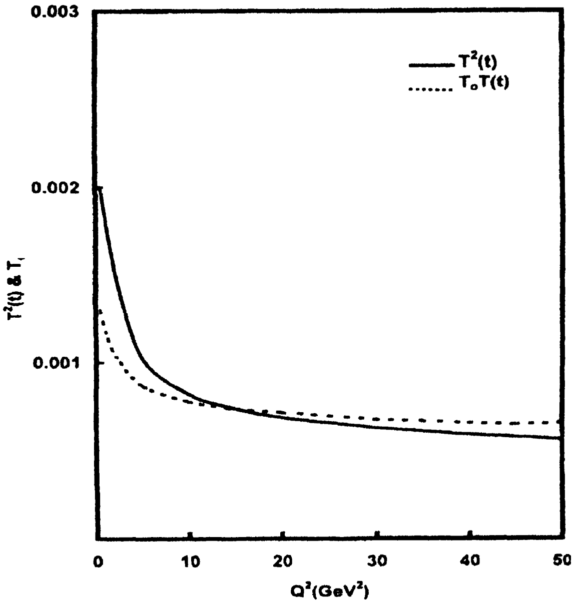

Figure 6. $T(t)^2$ (solid line) and $T_0T(t)$ (dashed line), where $T(t) = \alpha_s/2\pi$ against Q^2 in the Q^2 range $0.5 \le Q^2 \le 50$ GeV^2. We observed that for $T_0 = 0.027$, error becomes minimum in the Q^2 range $0.5 \le Q^2 \le 50$ GeV^2.

From our above discussion, it has been observed that we can not establish a unique relation between singlet and gluon structure functions *i.e.* a unique expression for $K(x)$ in eq. (9) by this method; $K(x)$ in the forms of an exponential function of x or a power in x can equally produce required x-distribution of deuteron structure functions. But unlike x-distribution function with many input parameters (generally used in the literature), our method required only one or two such parameters. The explicit form of $K(x)$ can actually be obtained only by solving coupled DGLAP evolution equations for singlet and gluon structure functions, and work is going on in this regard. Traditionally, the DGLAP equations provide a means of calculating the manner in which the parton distributions change at fixed x as Q^2 varies. This change comes about because of the various types of parton branching emission processes and the x-distributions are modified as the initial momentum is shared among the various daughter partons. However, the exact rate of modifications of x-distributions at fixed Q^2can not be obtained from the DGLAP equations, since it depends not only on the initial x but also on the rate of change of parton distributions with respect to x, $d^n F(x) / dx^n$ ($n = 1$ to ∞), up to infinite order. Physically, this implies that at high-x, the parton has a large momentum fraction at its disposal and as a result, it radiates partons including gluons in innumerable ways, some of them involving complicated QCD mechanisms. However, for low-x, many of the radiation processes will cease to occur due to momentum constraints and the x-evolutions get simplified. It is then possible to visualize a situation in which the modification of the x-distribution simply depends on its initial value and its first derivative. In this simplified situation, the DGLAP equations give information on the shapes of the x-distribution as demonstrated in this paper. The clearer testing of our results of x-evolution is actually the eq. (23) which is free from the additional assumption [eq. (9)]. The required non-singlet data is not adequately available in the low-x region to test our result.

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